Wormholes and Time-Machines

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Abstract

It has been proposed that wormholes can be made to function as time-machines. This opens up the question of whether this can be accomodated within a self-consistent physics or not. In this contribution we present some quantum mechanical considerations in this respect.

1 Introduction

Consider a flat space-time in which two regions (mouth A and mouth B) are connected by a throught (a wormhole) and assume that the intrinsic length of this wormhole is small compared to the distance between mouth A and mouth B in the flat space-time. It can be shown that this geometry can lead to the existence of closed time-like curves either by, at some period of time, accelerating one mouth of the wormhole relative to the other [1,3–5] or by placing the mouths in regions of differing gravitational potentials. An observer on a closed time-like curve can influence not only his own future but also his past; the wormhole thus can be made to function as a time machine.

The problem with which we are especially concerned is the "quantum billiard problem," the quantum analogue of the classical problem proposed by Novikov et al. [2,4,13] The idea, in the classical case, is the following, consider a wormhole acting like a time machine, denote the two mouths by A and B respectively. We can assume that an object by traveling from A to B moves back in time by a certain (fixed) timestep T (as seen from the external essentially flat space-time). Now suppose

we let a billiard ball roll towards mouth A, it will then go through the wormhole and "reappear" in the past at mouth B. Now, this gives the possibility that the ball exiting from mouth B hits the original ball. Unless this scattering is sufficiently weak, the original ball will be scattered off its original course and may not enter mouth A, in which case it obviously cannot go back in time, and exit from B just in time to hit itself off course, and thus it *will* reach mouth A... And we have a paradox. Novikov's idea was to allow only self-consistent solutions, i.e. the ball exiting from mouth B is allowed to hit the original ball on its way to mouth A, provided that it does not lead to a radically new trajectory for the original ball. Whatever happens, the original ball *must* reach mouth A. It has been proven that such selfconsistent solutions do exist classically. The question is, do they also exist quantum mechanically? In the quantum billiard problem, in an attempt to answer this, we consider wave packets instead of balls, and these interact through some potential. Pertubation theory then gives us a selfconsistency requirement, which we can actually solve in a few cases.

A related problem is the question of unitarity. In order to study this we must write down a model Hamiltonian and calculate the evolution operator. For reasons of space, this line of attack will not be followed here, instead I just refer to [16].

This seems to us to be the obvious way of attempting to answer these questions, but unfortunately it is not without problems. A space-time possesing closed time-like curves is not foliable, and hence we cannot, *in the vicinity of the wormhole*, make the 3+1 splitting of space-time essential to Schrödinger mechanics. This problem can be overcome when, instead of including the wormhole directly, we include it only in an effective theory, in such a way that we do not have to use Schrödinger theory near the wormhole but only sufficiently far away from it where we empirically know ordinary quantum theory to be correct. In this effective theory, the possibility of going through the wormhole and backwards in time gives rise to an interaction which looks much like an ordinary self-energy diagram. Away from the wormhole, Schrödinger mechanics must be valid but we have to take the self-interactions introduced by the presence of a time machine into account.

To keep things so simple that an analytical solution to the problem is possible we will assume that the only effect of the wormhole on the

wave packet, besides moving it to a different place in space-time, is the possibility of a shift in momentum. The diverging lense effect of the wormhole [4] will be ignored, as will the scattering of the wave upon the wormhole mouths. As we are thus only interested in the possibility or impossibility of selfconsistent motion we think that this crude model of the wormhole's interaction with its surroundings should suffice. Any quantum mechanical model taking the wormhole (and the resulting absence of a foliation) into considerations has to be equivalent to the Schrödinger theory sufficiently far away, where space-time is supposed to be flat. Hence sufficiently far away, any model has to be equivalent to ours, although perhaps with a different scattering kernel. If not, the mere applicability of Schrödinger quantum theory today would exclude the existence, anywhere in the universe, of regions with closed time-like curves.

2 The Selfconsistency Requirement for a Non-Relativistic Wave-Packet with Coulomb Interactions in 2+1 Dimensions

The simplest nom-trivial spatial dimension is two, and for simplicty we will restrict ourselves to that; the results, however, would not be radically different in higher dimensions.

From non-relativistic quantum mechanics, we know that a wave-packet ψ_i can undergo a transition $\psi_i \to \psi_f$ in the presence of a pertubation. To first order, pertubation theory gives

$$\psi_f(\mathbf{k}') = \int \frac{V_{\mathbf{k}\mathbf{k}'}}{\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}} \psi_i(\mathbf{k}) d^2k \tag{1}$$

where $V_{\mathbf{k}\mathbf{k}'} = \langle \mathbf{k}' | V | \mathbf{k} \rangle$ is the matrix element of the pertubation and where $\epsilon_{\mathbf{k}}$ is the energy, $\epsilon_{\mathbf{k}} = k^2/2m$. We will choose units in which $m = 1$.

The case of the quantum billiard problem is quite special: The potential in which ψ_i scatters is derived from its future "self", ψ_f, which gives rise to a "charge distribution" $\rho(\mathbf{x}, t) = |\psi_f(\mathbf{x}, t)|^2$, i.e. $V = V[\psi_f]$. We

will choose a potential of the form[1]

$$V(r) = \alpha' \rho(\mathbf{x},t) r^{\epsilon-1} = \alpha' |\psi_f(\mathbf{x},t))|^2 r^{\epsilon-1} \equiv v(r)|\psi_f(\mathbf{x},t)|^2 \tag{2}$$

where ϵ is taken to be small. Denoting the Fourier coefficients of the original state by $a_{\mathbf{k}}$, and those of the scattered state (that which goes through the wormhole and scatters its former "self") by $c_{\mathbf{k}}$

$$\begin{aligned} \psi_i(\mathbf{x}) &= \frac{1}{(2\pi)^{n/2}} \int a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} d^2x \\ \psi_f(\mathbf{x}) &= \frac{1}{(2\pi)^{n/2}} \int c_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} d^2x \end{aligned}$$

we obtain upon insertion in eq(1), by using the Fourier convolution theorem twice and changing variables a couple of times

$$\begin{aligned} c_{\mathbf{k}'} &= \alpha' \int \frac{a_{\mathbf{k}}}{\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}} c^*_{\mathbf{l}} c_{\mathbf{q}-\mathbf{l}} \|\mathbf{k} - \mathbf{k}' - \mathbf{q}\|^{\epsilon-2} d^2q d^2l d^2k && (3) \\ &\equiv \int c^*_{\mathbf{p}} \hat{X}^{\mathbf{p},\mathbf{q}}_{\mathbf{k}'} c_{\mathbf{q}} d^2p d^2q && (4) \end{aligned}$$

where we have introduced a scattering kernel

$$\begin{aligned} \hat{X}^{\mathbf{p},\mathbf{q}}_{\mathbf{k}'} &\equiv \int \frac{a_{\mathbf{k}} \tilde{v}(\mathbf{l})}{\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}+\mathbf{l}+\mathbf{p}+\mathbf{q}}} d^2k d^2l && (5) \\ &= 2 \int \frac{a_{\mathbf{k}} \tilde{v}(\mathbf{l})}{(\mathbf{k}')^2 - (\mathbf{k}+\mathbf{l}+\mathbf{p}+\mathbf{q})^2} d^2k d^2l && (6) \end{aligned}$$

where $\tilde{v}(k) = \alpha' \|k\|^{\epsilon-2}$ is the Fourier transform of $v(r) = \alpha' r^{\epsilon-1}$ and where we have inserted $\epsilon_{\mathbf{k}} = \frac{1}{2}\mathbf{k}^2$. To find the self-consistent solutions we must solve the (infinite dimensional) quadratic equation (4). In a more general set-up the kernel would also contain information about the structure and geometry of the wormhole. Thus it is essentially this quantity which hides our ignorance of the detailed structure of the wormhole.

[1]One must either choose this prescription for an almost Coulomb potential or screen the Coulomb potential in order to obtain a finite theory. Some comments on the case of the screened potential (the Yukawa potential) will be made at the end of this section.

2.1 On the Consistency and Limitations of this Formalism

Now, as mentioned in the introduction, space-times with closed time-like curves do not admit a foliation, and hence ordinary quantum mechanics is in principle meaningless. Thus, some comments on the consistency of the proposed formalism are in order. First of all, we do not attempt to give a quantum description of the dynamics close to the wormhole mouths. Sufficiently far outside that region, where space-time is almost flat, foliations are possible and ordinary quantum mechanics is known to be valid. But we are interested in the interaction of this "forbidden region" with its surroundings, as we want the wave-packet to traverse the wormhole. We then try to set-up an *effective* theory, which can accomodate this. The "forbidden region" is considered as a kind of "black box", which interacts with the environment: particles can enter it, and it emits particles too. This would be completely analogous to the situation of a quantum mechanical system interacting with a classical system, were it not for the added feature of special (temporal) correlations. A particle entering the region at time t is correlated with a particle exiting at the earlier time $t-T$, where T is the typical time-step of the wormhole. If the original wave packet is to interact with it we can desrcibe this in terms of an effective interaction $V = v(r)|\psi_f(\mathbf{x},t)|^2$, where $v(r)$ is the potential between two classical point particles and $|\psi_f|^2$ is the charge-distribution. We can reexpress the potential in terms of the initial wave-packet by writing $|\psi_f(\mathbf{x},t)|^2 = w(\mathbf{x},t)|\psi_i(\mathbf{x},t)|^2$ whereby the effective potential becomes $V(\mathbf{x},t) = w(\mathbf{x},t)|\psi_i(\mathbf{x},t)|^2$. This holds provided we stay away from the at most a countable number of zeroes of ψ_i, which thus form a set of measure zero. Simply plugging this into a Schrödinger equation leads to the following effective equation of motion (dropping the subscript i on ψ):

$$-\frac{1}{2}\nabla^2\psi + w(\mathbf{x},t)|\psi|^2\psi = i\frac{\partial}{\partial t}\psi \tag{7}$$

which is a slight generalization of the well-known non-linear Schrödinger equation [14], the only new feature being the non-constant coefficient $w(\mathbf{x},t)$.

Giving up describing the dynamics inside the "forbidden region", we can essentially use ordinary Schrödinger mechanics outside, but with an

effective potential depending upon the wave function, thus leading to a generalization of the non-linear Schrödinger equation as the effective equation of motion. Hence, as an effective theory, the proposed formalism should suffice. Thus also, in principle, one could calculate all sorts of transition amplitudes using the scattering theory of this generalization of the non-linear Schrödinger equation.[2]

2.2 Gaussian Wave Packets in an Almost Coulomb Potential

With a Gaussian wave-packet (the prototype of a localized wave-packet, i.e. the best analogue of a classical object) as our initial wave function, parameterized as

$$a_{\mathbf{k}} = e^{-ak^2+\mathbf{b}\cdot\mathbf{k}+c} \tag{8}$$

we obtain after a lenghty and tedious but standard calculation:

$$\hat{X}^{\mathbf{k}'}_{\mathbf{pq}} = 2\alpha'\pi^2 B(\epsilon, 1-\epsilon)\int_0^\infty e^{-ax^2+c'} I_0(b'x)\frac{x}{(k'^2-x^2)^{1-\epsilon}}dx \tag{9}$$

where $B(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function, I_0 is a modified Bessel function, and the coefficients are

$$\begin{aligned} b' &= \|\mathbf{b} - 2a(\mathbf{p}-\mathbf{q})\| && (10)\\ c' &= c - a(\mathbf{p}+\mathbf{q})^2 - \mathbf{b}\cdot(\mathbf{p}+\mathbf{q}) && (11)\end{aligned}$$

Proceed from eq(9) by noting that the Beta function, B, is singular in the limit $\epsilon \longrightarrow 0$. This comes as no surprise as this is the place where we put the infinities arising from the nature of the Coulomb field. Imagining the Beta function removed (or rather regularized) by renormalization makes it plausible to put $\epsilon \equiv 0$ inside the integral and simply treat the Beta function taken at $\epsilon = 0$ as a (finite) constant. Doing this we can evaluate the above integral obtaining

$$\hat{X}^{\mathbf{k}'}_{\mathbf{pq}} = 2i\alpha'\pi^3 B(\epsilon, 1-\epsilon)e^{-a(k')^2+c'} I_0(b'k') \tag{12}$$

[2]We are of course aware of the somewhat hand-waving charachter of these arguments, but we do believe they can be made more rigorous with very little effort. For simplicity, however, we have decided to stick to this simplistic argument.

Note from eqs(4),(12) that $c_{\mathbf{k}'}$ will always go like a Gaussian times some function, i.e.

$$c_{\mathbf{p}} = f(\mathbf{p})e^{-\alpha p^2} \tag{13}$$

The selfconsistency condition then reads

$$\begin{aligned} c_{\mathbf{k}'} = \quad & 2\pi\xi e^{-ak'^2}\sum_{nm} b_n b_m \int_0^\infty (p_+^2 - p_-^2)^{\frac{n+m}{2}} P_{\frac{n+m}{2}}\left(\frac{p_+^2+p_-^2}{p_+^2-p_-^2}\right)\times \\ & 2^{(n+m)/2} I_0(Ap_-)e^{-\alpha p_-^2-(a+\alpha)p_+^2}p_+p_-dp_+dp_- \end{aligned} \tag{14}$$

where we have defined

$$A = 2^{3/4}\sqrt{a}k' \text{ and } \xi = 2i\alpha'\pi^3 B(\epsilon, 1-\epsilon)e^c \tag{15}$$

with $\mathbf{p}_\pm = 2^{-1/2}(\mathbf{p}\pm\mathbf{q})$ and where b_n denotes the Taylor coefficients of $f(\mathbf{p}) = \sum b_p \mathbf{p}^n$ (one can prove that only even powers of $\mathbf{p}$ can possibly satisfy the selfonsistency requirement). This integral can actually be carried out, and we arrive at

$$\begin{aligned} c_{\mathbf{k}'} &= e^{-\alpha\mathbf{k}'^2}\sum_n b_{2n}\mathbf{k}'^{2n} \\ &= 2\pi^2\xi e^{-ak'^2}\sum_{mn}\left[b_{2n}b_{2m}\sum_{kl}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} m \\ l \end{array}\right)2^{k+l+n+m}\frac{(k+l-1)!!}{(k+l)!!}\times\right. \\ &\left.\sum_{l'}\left(\begin{array}{c} n+m-k-l \\ l' \end{array}\right)\frac{\Gamma(\frac{n+m+2-l'}{2})}{\alpha^{(n+m+2-l')/2}}\Phi(\frac{n+m+2-l'}{2},1;\frac{A^2}{4\alpha})C_{k+l+l'+1}\right] \end{aligned} \tag{16}$$

where $\Phi(a,b;z)$ is a degenerate hypergeometric function [6]

$$\Phi(a,b;z) \equiv \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}$$

with $(a)_n \equiv a(a+1)(a+2)...(a+n-1)$, and where we have introduced

$$C_\nu \equiv \left\{\begin{array}{ll} \frac{(2\lambda-1)!!}{2(2(a+\alpha))^\lambda}\sqrt{\frac{\pi}{a+\alpha}} & \nu = 2\lambda \\ \frac{\lambda!}{2(a+\alpha)^{\lambda+1}} & \nu = 2\lambda+1 \end{array}\right. \tag{17}$$

Note that this is an exact result, no approximations have been made. This equation constitutes the *final form of the selfconsistency requirement* in the case of an incomming Gaussian wave packet interacting

through a Coulomb potential, and so it is this equation we have to solve to find self consistent solutions. This selfconsistency requirement can only be solved (in principle) in the two extreme cases: $f =$ constant and f not a polynomial (i.e. the Taylor series never terminates, in which case f would be some analytic function of k'^2). Due to the hypergeometric function on the right hand side, the selfconsistency equation has no solutions when f is a polynomial.

If the wave packet is a pure Gaussian, we get a requirement on the wave packet traversing the wormhole (remember that the α refers to ψ_f, whereas the a refers to ψ_i). By putting $c_{\mathbf{k}'} = b_0 \exp(-\alpha k'^2)$ (cf.eq(12)) on the left hand side and similarly on the right hand side, where only one term in the sum would then appear, one immediatly sees that

$$b_0 = b_0^2 2\pi^2 \xi \tag{18}$$

i.e. $b_0 = (2\pi^2\xi)^{-1}$ (or $b_0 = 0$, but this would correspond to $\psi \equiv 0$ and is hence not interesting). Also, by differentiating twice with respect to k' and putting $k' = 0$ one gets

$$\alpha = \frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 + 2\sqrt{2}a} \tag{19}$$

i.e. there are exactly two solutions for the scattered wave packet.

This case, where the wave-packet is Gaussian at all times (i.e. both before and after scattering) is the quantum analogue of the classical billiard problem (the Gaussian wave-packets which are substituted for the billiard balls are as localized as the Heisenberg uncertainty principle allows).

The normalization of the wave-packet introduces a requirement on the *original* wave packet (i.e. on a). Normalization of the Gaussian wave-packet would require $b_0 = \sqrt{\frac{2\alpha}{\pi}}$, i.e. in order to have normalized wave packets we would have to impose the requirement

$$\alpha = (8\pi^3\xi^2)^{-1} = -(64ia\pi^7\alpha'^2 B^2(\epsilon, 1-\epsilon))^{-1} \tag{20}$$

where we have inserted the definition of ξ (see eq(14), also, remember that α' is the coupling constant) and demanded that the original wave-packet is normalized (i.e. $e^c = \sqrt{2a/\pi}$). This then allows only one *original Gaussian*, namely that with a satisfying

$$a^2 \pm a\sqrt{a^2 + 2\sqrt{2}a} = -(32i\pi^7\alpha'^2 B^2(\epsilon, 1-\epsilon))^{-1} \tag{21}$$

Thus there are only two solutions; in units where the right hand side is equal to one, we find $a = 0.5337543$ when we use the plus sign and $a = 1.4799995$ when we use the minus sign.
Had we chosen a Yukawa potential, $v(r) = \alpha' r^{-1} e^{-r/m}$, instead we would have had to make the substitution $k'^2 \to k'^2 + m^2$ in all the expressions, and the coefficients would become non-singular in the limit $\epsilon \to 0$. Hence eq(12) would become

$$\hat{X}^{\mathbf{k'}}_{\mathbf{pq}} \propto e^{-a(k'^2+m^2)+c'} I_0(b'\sqrt{k'^2+m^2})$$

and the quantity A defined in eq(15) would become $A = 2^{3/4}\sqrt{a}\sqrt{k'^2+m^2}$. This would make finding solutions in the general case even more difficult, as the right hand side of the selfconsistency requirement, eq(16), now would contain terms of the form $(\sqrt{k'^2+m^2})^n$ where n is just some integer. This will make analytical solutions very difficult to find. The Gaussian solution would still exist, though, but with b_0 multiplied by $\exp(am^2)$. Similarly, the right hand side of eq(20) would also change, but can still be taken to unity by an appropriate choice of m and α'. One can furthermore also consider plane waves, these do not, however, correspond to classical billiard balls, and moreover, analytical solutions are much more difficult to find.

2.3 Stability of Solutions

In the preceding section we found solutions to the selfconsistency requirement for pure Gaussians. By analogy with the classical billiard problem, where one attempts to avoid the situation in which the scattering of the incoming ball on its future self makes the incoming ball fail to traverse the wormhole, we now want to investigate the stability of these solutions under small perturbations. Write the selfconsistency condition in a symbolic form as

$$c_{\mathbf{k'}} = \hat{X}^{\mathbf{p},\mathbf{q}}_{\mathbf{k'}} c_{\mathbf{p}} c_{\mathbf{q}} \tag{22}$$

invoking a generalized summation convention consisting in integrating over repeated indices. We then consider a slight pertubation $\delta c_{\mathbf{k}}$ of a fixed solution $\zeta_{\mathbf{k}}$, to first order in the pertubation we then get

$$\delta c_{\mathbf{k'}} = \hat{X}^{\mathbf{p},\mathbf{q}}_{\mathbf{k'}} \zeta_{\mathbf{p}} \delta c_{\mathbf{q}} + \hat{X}^{\mathbf{p},\mathbf{q}}_{\mathbf{k'}} \zeta_{\mathbf{q}} \delta c_{\mathbf{p}} \equiv \hat{M}^{\mathbf{l}}_{\mathbf{k'}} \delta c_{\mathbf{l}} \tag{23}$$

where

$$\hat{M}_{\mathbf{k}'}^{\ \mathbf{l}} \equiv \hat{X}_{\mathbf{k}'}^{\mathbf{p},\mathbf{q}} \left(\zeta_{\mathbf{p}} \delta_{\mathbf{q}}^{\ \mathbf{l}} + \zeta_{\mathbf{q}} \delta_{\mathbf{p}}^{\ \mathbf{l}}\right) \tag{24}$$

with $\delta_{\mathbf{k}}^{\ \mathbf{l}} \equiv \delta^{(n)}(\mathbf{k}-\mathbf{l})$.
To study the stability of the solution $\zeta_{\mathbf{p}}$ we must consider the infinite dimensional map

$$\delta c_{\mathbf{k}'}^{(n)} \mapsto \delta c_{\mathbf{k}'}^{(n+1)} = \hat{M}_{\mathbf{k}'}^{\ \mathbf{l}} \delta c_{\mathbf{l}}^{(n)} \tag{25}$$

The difference, $\Delta_{\mathbf{k}}^{(n)}$, between two such iterates is then simply

$$\Delta_{\mathbf{k}'}^{(n)} = |(\hat{M}_{\mathbf{k}'}^{\ \mathbf{l}} - \delta_{\mathbf{k}'}^{\ \mathbf{l}}) \delta c_{\mathbf{l}}^{(n)}| = |(\hat{M}_{\mathbf{k}'}^{\ \mathbf{l}} - \delta_{\mathbf{k}'}^{\ \mathbf{l}})^n \delta c_{\mathbf{l}}^{(0)}| \tag{26}$$

This difference then goes like the size of the eigenvalue of $\hat{M}_{\mathbf{k}'}^{\ \mathbf{l}}$ corresponding to $\mathbf{k}'$. Denoting this eigenvalue by $\lambda(\mathbf{k}')$ we get

$$\Delta_{\mathbf{k}'}^{(n)} \sim |(\lambda(\mathbf{k}') - 1)^n| |\delta c_{\mathbf{k}'}^{(0)}| \tag{27}$$

i.e. it diverges when $\lambda(\mathbf{k}') > 2$,in which case then, the solution is unstable against slight pertubations. If, on the other hand, $\lambda(\mathbf{k}') < 2$ then the corresponding solution $\zeta_{\mathbf{k}}$ is stable. In accordance with the language of chaos theory we call $\lambda(\mathbf{k}')$ the *generalized Lyapunov exponent*.[3]
We have reduced the problem of stability to that of finding the eigenvalues of the integral operator $\hat{M}_{\mathbf{k}}^{\ \mathbf{l}}$.
Now proceed to examine the pure Gaussian solution found in the previous section;

$$\zeta_{\mathbf{k}} = b_0 e^{-\alpha k^2} \qquad (= c_{\mathbf{k}}) \tag{28}$$

Inserting this into the definition of the operator $\hat{M}_{\mathbf{k}}^{\ \mathbf{l}}$ we get, by construction, essentialy the same integrals as those we performed in order to solve the selfconsistency requirement. Explicitly

$$\lambda(\mathbf{k}) = \left(b_0 e^{-ak^2} \left(\int I_0(k\|\mathbf{p}-\mathbf{l}\|) e^{-a(\mathbf{p}+\mathbf{l})^2 - \alpha p^2} d^2p + (\mathbf{p} \to \mathbf{q})\right)\right) \delta_{\mathbf{k}}^{\ \mathbf{l}} \tag{29}$$

We can get, by ignoring the terms linear in $\mathbf{p}, \mathbf{q}$ in the exponent, an approximate expression for these integrals

$$\lambda(\mathbf{k}) \approx 2b e^{-\left(2a + \frac{1}{8(a+\alpha)^2}\right)k^2} \sqrt{\frac{\pi}{a+\alpha}} I_0 \left(\frac{k^2}{8(a+\alpha)^2}\right) \tag{30}$$

[3]In the usual finite dimensional case treated in chaos theory, this exponential is a function of the solution $\vec{\zeta}$, which in our infinite dimensional analogue means that λ is a functional of $\zeta_{\mathbf{k}}$.

Into this we can then insert $b_0 = \sqrt{\pi/\alpha}$ and $\alpha = (a + \sqrt{a^2 + 2a\sqrt{2}})/2$. Doing this we find a stable ($\lambda < 2$) as well as an unstable ($\lambda > 2$) region. It turns out that the solution is unstable for small a, but gets more and more stable as a grows, i.e. as the *original* wave packet becomes more and more localized in momentum space. But this implies that the wave packets are very diffuse in position space, i.e. does not at all look like a classical point particle. In fact, the more the quantum nature is apparent i.e. the larger the uncertainty in position, the better the stability of the Gaussian solution. We conclude the selfconsistent solutions of the classical billiard problem of e.g. Novikov [2] in this quantum mechanical framework (localized wave-packets) are unstable.

3 Discussion & Conclusion

One should first of all notice that we did not use any knowledge of the wormhole, nor did we specify where the interactions take place; this just has to be sufficiently (depending on the size of the wormhole) far away from it, where space-time is flat. Only the momentum of the packets were specified, and hence their location is indeterminate – the wormhole simply effectively introduces a new (self) interaction.

We ignored any effect the traversal of the wormhole might have on the wave packet except for a possible shift in momentum. In particular we let the diverging lense effect [4] out of consideration and the scattering of the wave upon the mouths. This does not seem to us to be able to alter the conclusions of this paper, because the resulting smaller amplitude, and consequently smaller scattering, could be compensated by changing the geometry of the problem, i.e. by changing the distance between the 'out-mouth' and the region where scattering occurs. Also of course, the assumption that the wave packets be small as compared to the wormhole is essential to the calculations made.

We have also ignored the possibility of the wave packet going through the wormhole more than once and therefore getting a larger shift in time (but with smaller amplitude of the shifted wave, due to the diverging lense effect), because this would just alter the region in which the scattering occurs; an effect which could be compensated again by changing the geometry appropriately. By the same token, we let out of consideration the possibility of writing the wave after scattering as a superposition

of waves that have traversed the wormhole a different number of times. This problem probably could be treated in a second quantized version of the above model, but could also be seen as just going to higher orders in the perturbation expansion and probably would not change much – it should be equivalent to a proper path-integration approach with a suitable highly non-trivial measure, taking only self-consistent solutions into account.

Thus we almost completely ignored the wormhole, which was the reason why we could use a Hamiltonian formulation. We have only included the time-machine effect of the wormhole, and this in a rather indirect manner, through an effective potential and hence through an effective equation of motion. This equation of motion turned out to be essentially the non-linear Schrödinger equation.

We derived a general, closed equation expressing the requirement of selfconsistency. This equation could be solved exactly only in the case where the Fourier coefficents of the wave packet after scattering, i.e. the part of the wave packet traveling on the closed timelike curve, has the form $c_{\mathbf{k}'} = b_0 \exp(-\alpha k'^2)$. If the solution was normalized we found only one possible value of the width of the incoming wavepacket and that the corresponding solution was unstable in large parts of parameter space, so only fine-tuning of initial conditions could render physics selfconsistent in these parts of parameter space. This need for fine-tuning springs from the restrictions on the form of the wave-packet. If one threw away this requirement the need for fine-tuning, i.e. the restrictions imposed upon the initial conditions, would in all likelihood become very much less severe. On the other hand, this form-requirement was essential for a semi-classical picture of “billiard balls” self-interacting due to the presence of a time-machine.

A fuller discussion of these problems will be given in $[15-16]$.

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